

Martingale and Weak Solutions for a Stochastic Nonlocal Burgers Equation on Bounded Intervals

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Abstract

This work is about the existence of martingale solutions and weak solutions for a stochastic nonlocal Burgers equation on bounded intervals. The existence of a martingale solution is shown by using a Galerkin approximation, Prokhorov's theorem and Skorokhod's embedding theorem. The same Galerkin approximation also leads to the existence of weak solution for the corresponding deterministic nonlocal Burgers equation on a bounded domain.

Keywords: Anomalous diffusion; Itô's formula; Stochastic Burgers equation; Nonlocal Burgers equation; Prokhorov's theorem; Skorokhod's embedding theorem.

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1 Introduction

The Fokker-Plank equation for a stochastic differential equation with an additive Brownian motion (a Gaussian process) is a usual diffusion equation with Laplacian operator Δ . When the Brownian motion is replaced by a α -stable Lévy motion (a non-Gaussian process) L_t^α , $\alpha \in (0, 2)$, the Fokker-Plank equation becomes a nonlocal partial differential equation [1] with a nonlocal Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$. When the drift (or vector field) of the stochastic differential equation depends on the distribution of the system evolution, this nonlocal partial differential equation becomes nonlinear. Nonlocal Laplacian operator also appears in mathematical models for viscoelastic materials (e.g., Kelvin-Voigt model), certain heat transfer processes in fractal and disordered media, and fluid flows and acoustic propagation in porous media [7, 22, 24]. Interestingly, a nonlocal diffusion equation also arises in pricing derivative securities in financial markets ([7]).

We consider the following stochastic nonlocal Burgers equation

$$\begin{cases} du(t) = \left(-(-\Delta)^{\frac{\alpha}{2}} u - uu_x \right) dt + g(u) dW(t), & t > 0, x \in D, \\ u|_{D^c} = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where $D = (-1, 1)$ is an interval in \mathbb{R}^1 , $D^c = \mathbb{R}^1 \setminus D$, u_0 is an given initial datum, and $(-\Delta)^{\frac{\alpha}{2}}$ is the nonlocal Laplacian operator defined by the following Cauchy principal value integral

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = C_\alpha \int_{\mathbb{R}^1 \setminus \{0\}} \frac{u(x+y) - u(x)}{|y|^{1+\alpha}} dy, \quad 0 < \alpha < 2, \quad (1.2)$$

where C_α is a negative constant depending on α . The Wiener process W_t will be specified later.

Some existing works: Nonlocal Burgers' type equations (deterministic or stochastic) on the whole real line have been considered by a number of authors. For example, Biler et al. [3] studied the following nonlocal equation

$$u_t = -(-\Delta)^{\frac{\alpha}{2}} u - uu_x, \quad x \in \mathbb{R}^1, \quad (1.3)$$

and proved the existence of a unique weak solution $u \in L^\infty(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R}))$ for $\alpha \in (\frac{3}{2}, 2]$. They further ([4]-[6]) extended this result to the equations of the Lévy conservation laws, and obtained the asymptotic behavior of solutions with anomalous diffusion for $1 < \alpha < 2$. Bertini et al. [2] studied the Burgers equation perturbed by a white noise and proved the existence of solutions by using the Cole-Hopf transformation in the stochastic setting. Wu et al. [30], Shi and Wang [29], and Debbi [12] considered various solutions for a class of stochastic partial differential equations, including Burgers equation as a special case. We remark that, in the whole space, the operator $(-\Delta)^{\frac{\alpha}{2}}$ is similar to the Laplace operator $-\Delta$ because we can use the Fourier transform to deal with the two operators.

However, there are much few existing works for nonlocal Burgers' type equations (deterministic or stochastic) on bounded domains. Mohammed-Zhang [23] proved the existence of solutions of the stochastic Burgers equation on a bounded interval with Dirichlet boundary conditions and anticipating initial data by Malliavin calculus.

A cautious remark: In fact, there is another, but very different, kind of 'fractional Laplacian operator' $(-\Delta)^{\frac{\alpha}{2}}$ on bounded domains in the literature. It is defined as a Fourier series expansion, in terms of non-negative eigenvalues and the orthonormal basis formed by the corresponding eigenfunctions for $-\Delta$. This is similar to the textbook definition of a fractional power for a positive-definite symmetric matrix in linear algebra. For example, Debbi [11] considered the fractional stochastic Navier-Stokes equations on bounded domains with this fractional Laplacian operator. We remark that this fractional Laplacian operator is *different* from the nonlocal Laplacian operator (1.2) which we use here in this paper.

On a bounded domain the local Laplacian operator $-\Delta$ and the nonlocal Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$ have significant differences (see Section 2 below). Especially, the usual fractional Sobolev spaces and embedding inequalities will not be suitable in this context. More information about the nonlocal operator $(-\Delta)^{\frac{\alpha}{2}}$ on bounded domains, are in [13, 9, 17, 18].

For the nonlocal stochastic Burgers equation (1.1), there is no hope to use the nonlocal or anomalous diffusion $(-\Delta)^{\frac{\alpha}{2}}$ to dominate the convection uu_x . Thus the usual method [21] and factorization method [8] are difficult to apply here. We will adopt the method used in [16] to obtain the existence of martingale solution to (1.1). Because $(-\Delta)^{\frac{\alpha}{2}}$ is a nonlocal operator and the usual fractional Sobolev spaces [25] will not be suitable, we will introduce a new weighted nonlocal Sobolev space. We also remark that the Hardy-Littlewood-Sobolev inequality does not hold when α is larger than the spatial dimension (which is 1 in this paper). Moreover, we prove the existence of L^2 weak solution for the nonlocal Burgers equation (1.1).

The rest of this paper is organized as follows. In section 2, we will recall some results of nonlocal Sobolev spaces. Section 3 is concerned with the proof of the main result on the existence of martingale solution. In section 4, we will consider the nonlocal deterministic Burgers equations in a bounded domain.

2 Preliminaries

In this section, we first recall the definition of classical fractional Sobolev space and then define a nonlocal weighted Sobolev space. Finally, we discuss some differences between these two kinds of spaces, and highlight special properties for the nonlocal Sobolev spaces.

2.1 Classical fractional Sobolev spaces

For $s \in (0, 1)$ and $p \in [1, +\infty)$, we define

$$W^{s,p}(D) := \left\{ u \in L^p(D) : \frac{u(x) - u(y)}{|x - y|^{\frac{n}{p} + s}} \in L^p(D \times D) \right\};$$

i.e., an intermediary Banach space between $L^p(D)$ and $W^{1,p}(D)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(D)} := \left(\int_D |u(x)|^p dx + \int_D \int_D \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}, \quad (2.1)$$

where $D \subseteq \mathbb{R}^n$ is a bounded domain and the term

$$[u]_{W^{s,p}(D)} := \left(\int_D \int_D \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}$$

is the so-called Gagliardo (semi) norm of u . Now we have the following embedding inequalities.

Lemma 2.1 [25, Propositions 2.1 and 2.2] *Let $p \in [1, \infty)$ and $0 < s \leq s' < 1$. Let D be an open set in \mathbb{R}^n and $u : D \rightarrow \mathbb{R}^n$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(D)} \leq C \|u\|_{W^{s',p}(D)}$$

for some suitable positive constant $C = C(n, s, p) \geq 1$. In particular, $W^{s',p}(D) \subseteq W^{s,p}(D)$. Furthermore, if D is an open set in \mathbb{R}^n of class $C^{0,1}$ with boundary, then

$$\|u\|_{W^{s,p}(D)} \leq C \|u\|_{W^{1,p}(D)}$$

for some suitable positive constant $C = C(n, s, p) \geq 1$. In particular, $W^{1,p}(D) \subseteq W^{s,p}(D)$.

Lemma 2.2 [25, Theorems 6.7 and 8.2] (i) *Let $s \in (0, 1)$ and $p \in [1, \infty)$ be such that $sp < n$. Let $D \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n, p, s, D)$ such that, for any $u \in W^{s,p}(D)$, we have*

$$\|u\|_{L^q(D)} \leq C \|u\|_{W^{s,p}(D)}$$

for any $q \in [p, p^*]$; i.e., the space $W^{s,p}(D)$ is continuously embedded in $L^q(D)$ for any $q \in [p, p^*]$, $p^* = \frac{np}{n-sp}$.

If, in addition, D is bounded, then the space $W^{s,p}(D)$ is continuously embedded in $L^q(D)$ for any $q \in [1, p^*]$.

(ii) Let $D \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}(D)$ with no external cusps and let $p \in [1, \infty)$, $s \in (0, 1)$ be such that $sp > n$. Then, there exists $C = C(n, p, s, D)$ such that

$$\|u\|_{C^{0,\beta}(D)} \leq C \|u\|_{W^{s,p}(D)}$$

for any $u \in L^p(D)$ with $\beta := \frac{sp-n}{p}$.

For $s \in (0, 1)$ and $p \in [1, \infty)$, we say that an open set $D \subseteq \mathbb{R}^n$ is an *extension domain* for $W^{s,p}$ if there exists a positive constant $C = C(n, p, s, D)$ such that: for every function $u \in W^{s,p}(D)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ with $\tilde{u}(x) = u(x)$ for any $x \in D$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,p}(D)}$.

When $s > 1$ and but not an integer, we write $s = m + \sigma$, where m is an integer and $\sigma \in (0, 1)$. In this case the space $W^{s,p}(D)$ consists of those equivalence classes of functions $u \in W^{m,p}(D)$ whose distributional derivatives $D^\beta u$, with $|\beta| = m$, belong to $W^{\sigma,p}(D)$, namely

$$W^{s,p}(D) := \left\{ u \in W^{m,p}(D) : D^\beta u \in W^{\sigma,p}(D) \text{ for any } \beta \text{ s.t. } |\beta| = m \right\}.$$

This is a Banach space with norm

$$\|u\|_{W^{s,p}(D)} := \left(\|u\|_{W^{m,p}(D)}^p + \|D^\beta u\|_{W^{\sigma,p}(D)}^p \right)^{\frac{1}{p}}.$$

Clearly, if $s = m$ is an integer, the space $W^{s,p}(D)$ coincides with the usual Sobolev space $W^{m,p}(D)$. We remark that when $s > 1$, Lemmas 2.1 and 2.2 also hold.

2.2 Nonlocal Sobolev spaces

In this paper, we are concerned with the case with $p = 2$. In order to define the nonlocal Sobolev space, we first decompose the operator $(-\Delta)^{\frac{\alpha}{2}}$ into two components, and then examine it as a divergence operator. Assume that $D \subset \mathbb{R}^n$ is an open bounded domain.

Inspired by [25], we rewrite the nonlocal Laplacian operator as

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} u(x) &= C_\alpha \int_{\mathbb{R}^n \setminus \{0\}} \frac{u(x+y) - u(x)}{|y|^{n+\alpha}} dy \\ &= C'_\alpha \int_D \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy + C'_\alpha u(x) \int_{D^c} \frac{dy}{|x-y|^{n+\alpha}}. \end{aligned}$$

where $C'_\alpha = -C_\alpha$. Now we examine the term with integral over D^c . Denote the shortest distance to boundary by $\delta(x) = \text{dist}(x, \partial D)$ and the longest distance to boundary by $\varrho(x) = \sup_{y \in \partial D} \{\text{dist}(x, y), y \in \partial D\}$. Then, we have

$$B_{\varrho(x)}^c(x) := \mathbb{R}^n \setminus B_{\varrho(x)}(x) \subseteq D^c \subseteq \mathbb{R}^n \setminus B_{\delta(x)}(x) := B_{\delta(x)}^c(x),$$

where $B_r(x)$ denotes the sphere with radius r and centered at x . Thus,

$$\begin{aligned} \int_{D^c} \frac{dy}{|y-x|^{n+\alpha}} &\geq \int_{B_{\varrho(x)}^c(x)} \frac{dy}{|y-x|^{n+\alpha}} \\ &\geq \frac{C}{\varrho(x)^\alpha}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \int_{D^c} \frac{dy}{|y-x|^{n+\alpha}} &\leq \int_{B_{\delta(x)}^c(x)} \frac{dy}{|y-x|^{n+\alpha}} \\ &\leq \frac{C}{\delta(x)^\alpha}, \end{aligned} \tag{2.3}$$

where C is a positive constant. When $n = 1$ and $D = (-1, 1)$, we have the following exact expression

$$\int_{D^c} \frac{dy}{|y-x|^{n+\alpha}} = \frac{1}{\alpha} \left(\frac{1}{(1+x)^\alpha} + \frac{1}{(1-x)^\alpha} \right).$$

Following [13], we can get another representation of the operator. We first give a general formula. Given the vector mapping $\mathcal{V}(x, y)$, $\beta(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ with β antisymmetric, i.e., $\beta(x, y) = -\beta(y, x)$, the action of the *nonlocal divergence operator* \mathcal{D} on \mathcal{V} is defined in [14] as

$$\mathcal{D}(\mathcal{V})(x) := - \int_{\mathbb{R}^n} (\mathcal{V}(x, y) + \mathcal{V}(y, x)) \cdot \beta(x, y) dy, \quad \text{for } x \in \mathbb{R}^n,$$

where $\mathcal{D}(\mathcal{V}) : \mathbb{R}^n \rightarrow \mathbb{R}$.

Given the mapping $u(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, the adjoint operator \mathcal{D}^* corresponding to \mathcal{D} is the operator whose action on u is given by

$$\mathcal{D}^*(u)(x, y) = -(u(y) - u(x))\beta(x, y), \quad \text{for } x, y \in \mathbb{R}^n,$$

where $\mathcal{D}^*(u) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^k$.

If $\Theta(x, y) = \Theta(y, x)$ denotes a second-order tensor satisfying $\Theta = \Theta^T$, then we have

$$\mathcal{D}(\Theta \cdot \mathcal{D}^*u)(x) = -2 \int_{\mathbb{R}^n} (u(y) - u(x))\beta(x, y) \cdot (\Theta(x, y) \cdot \beta(x, y)) dx, \quad \text{for } x \in \mathbb{R}^n,$$

where $\mathcal{D}(\Theta \cdot \mathcal{D}^*u) : \mathbb{R}^n \rightarrow \mathbb{R}$. If we let Θ be the identity matrix, and β be such that

$$2|\beta(x, y)| = \frac{C'_\alpha}{|x-y|^{n+\alpha}},$$

then

$$\mathcal{D}(\Theta \cdot \mathcal{D}^*u)(x) = -C'_\alpha \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy,$$

where $\mathcal{D}^*(u)(x, y) = (u(x) - u(y)) \frac{x-y}{|x-y|^{\frac{n+\alpha}{2}+1}}$.

In particular, for $n = 1$, we obtain

$$(-\Delta)^{\frac{\alpha}{2}} u = C'_\alpha \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x-y|^{1+\alpha}} dy = -\mathcal{D}(\Theta \cdot \mathcal{D}^*u)(x),$$

that is, the operator $(-\Delta)^{\frac{\alpha}{2}}$ is a divergence operator. For simplicity, we will set $C'_\alpha = 1$.

Direct calculations lead to

$$\begin{aligned} ((-\Delta)^{\frac{\alpha}{2}} u, u)_{L^2(D)} &= -(\mathcal{D}(\mathcal{D}^*u)(x), u(x))_{L^2(D)} \\ &= -(\mathcal{D}(\mathcal{D}^*u)(x), u(x))_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} dy dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{2u^2(x)}{|x-y|^{n+\alpha}} dy dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{2u(x)u(y)}{|x-y|^{n+\alpha}} dy dx \\ &= 2 \int_D u^2(x) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n+\alpha}} dy dx + \int_D \int_D \frac{2u(x)u(y)}{|x-y|^{n+\alpha}} dy dx \\ &= \int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} dy dx + \int_D \int_{D^c} \frac{2u^2(x)}{|x-y|^{n+\alpha}} dy dx \\ &= [u]_{W^{\frac{\alpha}{2}, 2}(D)}^2 + \int_D \int_{D^c} \frac{2u^2(x)}{|x-y|^{n+\alpha}} dy dx, \end{aligned} \tag{2.4}$$

where we have used the fact that $u|_{D^c} = 0$. In particular, when $n = 1$ and $D = (-1, 1)$, we have

$$((-\Delta)^{\frac{\alpha}{2}}u, u)_{L^2(D)} = [u]_{W^{\frac{\alpha}{2}, 2}(D)}^2 + \frac{2}{\alpha} \int_D u^2(x) \left(\frac{1}{(1+x)^\alpha} + \frac{1}{(1-x)^\alpha} \right) dx.$$

We remark that $W^{\frac{\alpha}{2}, 2}(D)$ is a Hilbert space. Actually, a scalar product is

$$(u, v)_{W^{\frac{\alpha}{2}, 2}(D)} = \int_D u(x)v(x)dx + \int_D \int_D \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+\alpha}} dydx. \quad (2.5)$$

It follows from (2.4) that the definition of fractional Sobolev space is not suitable. The reason is that we cannot make sure that the term

$$\int_D \int_{D^c} \frac{2u^2(x)}{|x - y|^{n+\alpha}} dydx < \infty.$$

Therefore, we will introduce a weighted nonlocal Sobolev space $W_\rho^{s,p}(D)$, with $0 < s < 1$, $p \geq 1$, and a ‘weight’ function

$$\rho(x) = \int_{D^c} \frac{2}{|x - y|^{n+\alpha}} dy.$$

By (2.2) and (2.3), we see that $\rho(x)$ has strictly positive lower bound. When $n = 1$ and $D = (-1, 1)$, we have

$$\rho(x) = \frac{2}{\alpha} \left(\frac{1}{(1+x)^\alpha} + \frac{1}{(1-x)^\alpha} \right).$$

Define

$$\|u\|_{W_\rho^{s,p}(D)} := \left(\int_D \rho(x)|u(x)|^p dx + \int_D \int_D \frac{(u(x) - u(y))^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}.$$

It follows from [19] that $W_\rho^{s,p}(D)$ is a Banach space. From (2.4), we know that $\|\mathcal{D}^*u\|_{L^2(D)} = \|u\|_{W_\rho^{s,2}(D)}$. Corresponding to (2.5), we can define

$$W_\rho^{-s,2}(D) \langle u, v \rangle_{W_\rho^{s,2}(D)} = \int_D \rho(x)u(x)v(x)dx + \int_D \int_D \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} dydx,$$

where $W_\rho^{-s,p}(D)$ denotes the topological dual space of $W_\rho^{s,p}(D)$. We can verify that Lemmas 2.1 and 2.2 also hold for the weighted nonlocal Sobolev space introduced here. Additionally, the weighted nonlocal Sobolev space introduced here is consistent with the definition of solution to equation (1.1) in [13]. That is, for $s = \frac{\alpha}{2}$, we have

$$\|u\|_{W_\rho^{s,p}(D)}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(-\Delta)^{\frac{\alpha}{2}} u dx dy.$$

If we define

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(-\Delta)^{\frac{\alpha}{2}} u dx dy < \infty \right\},$$

then

$$W_\rho^{s,p}(D) = \{u \in H^s(\mathbb{R}^n), u \equiv 0 \text{ in } \mathbb{R}^n \setminus D\}.$$

Thus a weighted nonlocal Sobolev space is defined. It is known that

$$\|u\|_{W_\rho^{-s,p}(D)} = \sup_{v \in W_\rho^{s,p}(D), \|v\|_{W_\rho^{s,p}(D)} \leq 1} W_\rho^{-s,p}(D) \langle u, v \rangle_{W_\rho^{s,p}(D)}.$$

Similar to (2.4), we have

$$\begin{aligned} W_\rho^{-\frac{\alpha}{2},2}(D) \langle (-\Delta)^{\frac{\alpha}{2}} u, v \rangle_{W_\rho^{\frac{\alpha}{2},2}(D)} &= \langle (-\Delta)^{\frac{\alpha}{2}} u, v \rangle_{L^2(D)} \\ &\leq \| \mathcal{D}^* u \|_{L^2(D)} \| \mathcal{D}^* v \|_{L^2(D)} \\ &= \| u \|_{W_\rho^{\frac{\alpha}{2},2}} \| v \|_{W_\rho^{\frac{\alpha}{2},2}}, \end{aligned}$$

which implies that

$$\| (-\Delta)^{\frac{\alpha}{2}} u \|_{W_\rho^{-\frac{\alpha}{2},2}(D)} \leq C \| u \|_{W_\rho^{\frac{\alpha}{2},2}(D)}. \quad (2.6)$$

When $s > 1$ and $s = m + \sigma$ with $\sigma \in (0, 1)$, we define

$$W_\rho^{s,p}(D) = \{u \in W_\rho^{m,p}(D), \mathbb{D}^\gamma u \in W_\rho^{\sigma,p}(D), |\gamma| = m\}$$

with the norm

$$\|u\|_{W_\rho^{s,p}(D)} = \|u\|_{W_\rho^{\sigma,p}(D)} + \left(\sum_{k \leq m} \int_D \rho(x) |\mathbb{D}^k u(x)|^p dx \right)^{\frac{1}{p}}.$$

Before we end this section, we present the following remark, which shows the difference between the classical Sobolev space and nonlocal Sobolev space, that is, a difference between local operators and nonlocal operators.

Remark 2.1 1. From the above definitions of two kinds of Sobolev spaces, it is clear that $W_\rho^{s,p}(D) \subset W^{s,p}(D)$. In particular, it follows from Lemma 2.1 that

$$W_\rho^{\frac{\alpha}{2},2}(D) \subset W^{\frac{\alpha}{2},2}(D) \subset L^2(D).$$

2. For $u \in W^{s,p}(D)$, we do not know any information about the function u on the boundary. Even if we consider the space $W_0^{s,p}(D)$, where $W_0^{s,p}(D) = \{u \in W^{s,p}(D) : u|_{\partial D} = 0\}$, we only know that the function $u = 0$ on the boundary and we do not know how the function u becomes 0. For example, consider the following problem

$$\begin{cases} -\Delta u = f(u), & \text{in } D, \\ u = 0, & \text{on } \partial D. \end{cases}$$

A working space for this problem is $W_0^{1,2}(D)$. Since $-\Delta$ is a local operator, we do not know how the solution $u(x)$ becomes 0 when $x \rightarrow \partial D$. However, in problem (1.1), the operator $(-\Delta)^{\frac{\alpha}{2}}$ is a nonlocal operator, that is, it is defined in the whole space. So, it has information about how u becomes 0 as $x \rightarrow \partial D$. In fact, from the definition of nonlocal Sobolev space, we know that $\frac{u(x)}{\delta(x)} \rightarrow 0$ as $x \rightarrow \partial D$, which dictates how u becomes 0 near boundary. It coincides with the result of Theorem 1.2 in [26] or also [27]. This is a significant difference between the fractional and nonlocal Sobolev spaces.

3. When D is the whole space \mathbb{R}^n , that is, $D^c = \emptyset$, then (2.4) becomes

$$((-\Delta)^{\frac{\alpha}{2}} u, u)_{L^2(D)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\alpha}} dy dx,$$

which coincides with the definition of $H^s(\mathbb{R}^n)$. Therefore, our definition of nonlocal Sobolev space is quite natural.

3 Martingale solution for a stochastic nonlocal Burgers equation

In this section we consider martingale solution for the stochastic nonlocal Burgers equation (1.1).

Let $W(t)$ be a Wiener process defined on a certain complete probability space (Ω, \mathcal{F}, P) and take values in the separable Hilbert space U , with incremental covariance operator Q . Let $(\mathcal{F}_t)_{t \geq 0}$ be the σ -algebras generated by $\{W(s), 0 \leq s \leq t\}$, then $W(t)$ is a martingale relative to $(\mathcal{F}_t)_{t \geq 0}$ and we have the following representation of $W(t)$:

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i,$$

where $\{e_i\}_{i \geq 1}$ is an orthonormal set of eigenvectors of Q , $\beta_i(t)$ are mutually independent real Wiener processes with incremental covariance $\lambda_i > 0$, $Qe_i = \lambda_i e_i$ and $\text{Tr} Q := \sum_{i=1}^{\infty} \lambda_i < \infty$. For an operator $G \in L_2(U, H)$, the space of all bounded linear operators from U into H , we denote by $\|G\|_2$ its Hilbert-Schmidt norm, i.e.,

$$\|G\|_2^2 := \text{Tr}(GQG^*).$$

Throughout this paper, we assume that $V = W_{\rho^{\frac{\alpha}{2}, 2}}(D)$, $H = L^2(D)$ and $V_1 = W_{\rho^{-\delta, 2}}(D)$, $\delta > 2 + \alpha$. Then we have

$$V \subseteq H = H^* \subseteq V^* := W_{\rho^{-\frac{\alpha}{2}, 2}}(D) \subseteq V_1.$$

In addition, we make the following assumption.

(C) The noise intensity $g : H \rightarrow L_2(U, H)$ is continuous and

$$\begin{aligned} \|g(u)\|_{L_2(U, H)}^2 &\leq C\|u\|_H^2 + \lambda, \\ \|g(u) - g(v)\|_{L_2(U, H)}^2 &\leq C\|u - v\|_H^2, \quad \forall u, v \in H \end{aligned}$$

for some positive real numbers C and λ . Here and hereafter, we assume C is a positive constant and may be different from line to line.

Definition 3.1 *We say that there exists a martingale solution of the equation (1.1) if there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, a Wiener process W on the space U and progressively measurable process $u : [0, T] \times \Omega \rightarrow H$, with \mathbb{P} -a.e. paths*

$$u(\cdot, \omega) \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap C([0, T]; V_1),$$

such that \mathbb{P} -a.e., the identity

$$\begin{aligned} (u, v)_H + \int_0^t {}_{V^*} \langle (-\Delta)^{\frac{\alpha}{2}} u, v \rangle_V ds + \int_0^t {}_{V^*} \langle uu_x, v \rangle_V ds \\ = (u_0, v)_H + {}_{V^*} \langle \int_0^t g(u) dW_s, v \rangle_V \end{aligned} \quad (3.1)$$

holds for all $t \in [0, T]$ and all $v \in V$.

The main result in this section is the following theorem.

Theorem 3.1 *Assume that $\alpha \in (1, 2)$ and u_0 be in $L^p(\Omega \rightarrow H; \mathcal{F}_0; \mathbb{P})$ for some $p \geq 4$. Then, under the assumption (C), there exists a martingale solution for the system (1.1).*

We now prove Theorem 3.1. The main ingredients are Galerkin approximations, Skorohod embedding theorem and the representation Theorem.

We divide the proof into 3 Steps.

Step 1. Finite-dimensional approximation

It follows from [20] that the operator $(-\Delta)^{\frac{\alpha}{2}}$ is positive and selfadjoint, with compact resolvent. We denote by $0 < \lambda_1 < \lambda_2 \leq \dots$ the eigenvalues of $(-\Delta)^{\frac{\alpha}{2}}$, and by ϕ_1, ϕ_2, \dots a complete orthonormal basis for H , formed by the corresponding eigenvectors. Let

$$\{e_1, e_2, \dots\} \subset V$$

be an orthonormal basis of H and let $H_n := \text{span}\{e_1, \dots, e_n\}$ such that $\{e_1, e_2, \dots\}$ is dense in V . Let $P_n : V^* \rightarrow H_n$ be defined by

$$P_n y := \sum_{i=1}^n \langle y, e_i \rangle e_i, \quad y \in V^*.$$

Obviously, $P_n|_H$ is just the orthogonal projection onto H_n in H and we have

$${}_V \langle P_n (-\Delta)^{\frac{\alpha}{2}} u, v \rangle_V = \langle P_n (-\Delta)^{\frac{\alpha}{2}} u, v \rangle_H = {}_{V^*} \langle (-\Delta)^{\frac{\alpha}{2}} u, v \rangle_V, \quad u \in V, v \in H_n,$$

where ${}_V \langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^* . In the following section of the paper, we will omit the subscript. Let $\{g_1, g_2, \dots\}$ be an orthonormal basis of U and

$$W^n(t) := \sum_{i=1}^n \langle W(t), g_i \rangle g_i = \tilde{P}_n W(t),$$

where \tilde{P}_n is the orthogonal projection onto $\text{span}\{g_1, \dots, g_n\}$ in U .

Then for each finite $n \in N$, we consider the following stochastic equation on H_n

$$\begin{cases} du^n(t) = (-P_n(-\Delta)^{\alpha/2} u^n(t) + P_n(u^n u_x^n)) dt + P_n g(u^n) dW^n(t), & t \in [0, T], \\ u^n(0) = P_n u_0 = u_0^n. \end{cases} \quad (3.2)$$

Since the finite dimensional space stochastic differential equation (3.2) has locally Lipschitz and linear growth coefficient, the equation (3.2) admits a unique strong solution $(u^n(t) \in L^2(\Omega; C([0, T]; H_n)))$, see [28] for the details.

Step 2. A priori estimate

By Itô formula and noting $((-\Delta)^{\alpha/2} u, u) = \|u\|_V^2$, $(u u_x, u) = 0$, we have

$$\begin{aligned} \|u^n(t)\|_H^2 &= \|u^n(0)\|_H^2 + 2 \int_0^t \langle -P_n(-\Delta)^{\alpha/2} u^n + P_n(u^n u_x^n), u^n \rangle ds \\ &\quad + 2 \int_0^t \langle u^n, P_n g(u^n) dW^n(s) \rangle + \int_0^t \|P_n g(u^n) \tilde{P}_n\|_2^2 ds \\ &= \|u^n(0)\|_H^2 - 2 \int_0^t \langle (-\Delta)^{\alpha/2} u^n, u^n \rangle ds \\ &\quad + 2 \int_0^t \langle u^n, g(u^n) dW^n(s) \rangle + \int_0^t \|P_n g(u^n) \tilde{P}_n\|_2^2 ds \\ &\leq \|u^n(0)\|_H^2 - 2 \int_0^t \|u^n\|_V^2 ds \\ &\quad + 2 \int_0^t \langle u^n, g(u^n) dW^n(s) \rangle + C \int_0^t \|u^n(s)\|_H^2 ds + \lambda T, \end{aligned}$$

which, after taking expectations, yields that

$$E\|u^n(t)\|_H^2 + 2E \int_0^t \|u^n\|_V^2 ds \leq E\|u^n(0)\|_H^2 + C \int_0^t \|u^n\|_H^2 ds + \lambda T.$$

By Gronwall's inequality, we have

$$\begin{aligned} \sup_{0 \leq t \leq T} E\|u^n(t)\|_H^2 &\leq c_1, \\ E \int_0^T \|u^n\|_V^2 ds &\leq c_2, \end{aligned} \quad (3.3)$$

where c_1, c_2 are positive constants.

On the other hand, by Itô formula and Yong's inequality, we obtain for $q \geq 2$

$$\begin{aligned} \|u^n(t)\|_H^q &= \|u^n(0)\|_H^q + q(q-2) \int_0^t \|u^n\|_H^{q-4} \|(P_n g(u^n) \tilde{P}_n)^* u^n(s)\|_H^2 ds \\ &\quad + \frac{q}{2} \int_0^t \|u^n\|_H^{q-2} \left(2\langle -P_n(-\Delta)^{\alpha/2} u^n + P_n(u^n u_x^n), u^n \rangle + \|P_n g(u^n) \tilde{P}_n\|_H^2 \right) ds \\ &\quad + q \int_0^t \|u^n\|_H^{q-2} \langle u^n, P_n g(u^n) dW^n(s) \rangle \\ &\leq \|u^n(0)\|_H^q - \frac{q\theta}{2} \int_0^t \|u^n\|_H^{q-2} \|u^n\|_V^2 ds + q(q - \frac{3}{2}) \int_0^t (\lambda \|u^n\|_H^{q-2} + C \|u^n\|_H^q) ds \\ &\quad + q \int_0^t \|u^n\|_H^{q-2} \langle u^n, P_n g(u^n) dW^n(s) \rangle. \end{aligned} \quad (3.4)$$

It follows from Burkholder-Davis-Gundy's inequality that

$$\begin{aligned} &q\mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \|u^n\|_H^{q-2} \langle u^n, P_n g(u^n) dW(s) \rangle \right| \right] \\ &\leq 3q\mathbb{E} \left[\left(\int_0^T \|u^n\|_H^{2q-2} \|g(u_s^n)\|_2^2 ds \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{3} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u^n(t)\|_H^q \right] + C \left(\int_0^T \mathbb{E} \|g(u_s^n)\|_2^2 ds \right)^{q/2} \\ &\leq \frac{1}{3} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u^n(t)\|_H^q \right] + C \left(\int_0^T \mathbb{E} \|u_s^n\|_{L_H^2}^2 ds \right)^{q/2} \\ &\leq \frac{1}{3} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u^n(t)\|_H^q \right] + CE \int_0^T \|u^n\|_H^q ds + C\lambda^q T, \end{aligned} \quad (3.5)$$

where we have used Yong's inequality. By (3.4), (3.5) and using Gronwall's inequality, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u^n(t)\|_H^q + \mathbb{E} \int_0^T \|u^n\|_H^{q-2} \|u^n\|_V^2 ds \leq C (\mathbb{E} \|u^n(0)\|_H^q + \lambda^q T), \quad (3.6)$$

where C does not depend on n .

Inspired by [11, 16], we have the following lemma.

Lemma 3.1 *The sequence $\{u^n\}_{n=1,2,\dots}$ of solutions of equation (3.2) is uniformly bounded in the space*

$$L^2(\Omega, W^{\gamma,2}(0, T; W_\rho^{-\delta,2}(D))) \cap L^2(0, T; V),$$

where $\delta < 2 + \alpha$ and $0 < \gamma < \frac{1}{2}$.

Proof. Inequality (3.3) implies that $\{u^n\}_{n=1,2,\dots}$ is uniformly bounded in $L^2(0, T; V)$. Now we prove another part. We recall that the Besov-Slobodetski space $W^{\gamma,p}(0, T; H)$ with H being a Banach space, $\gamma \in (0, 1)$ and $p \geq 1$, is the space of all $v \in L^p(0, T; H)$ such that

$$\|u\|_{W^{\gamma,p}(0,T;H)} := \left(\int_0^T \|u(t)\|_H^p dt + \int_0^T \int_0^T \frac{\|u(x) - u(y)\|_H^p}{|t - s|^{1+\gamma p}} dt ds \right)^{\frac{1}{p}},$$

As $\{u^n(t)\}_{t \in [0, T]}$ is the strong solution of the finite dimensional stochastic differential equation (3.2), then $u^n(t)$ is the solution of the stochastic integral equation

$$u^n(t) = P_n u_0 + \int_0^t (-(\Delta)^{\alpha/2} u^n(s) + u^n(s) u_x^n(s)) ds + \int_0^t P_n g(u^n) dW^n(s), \quad a.s.$$

for all $t \in [0, T]$. We denote by

$$\begin{aligned} I_1(t) &= \int_0^t (-(\Delta)^{\alpha/2} u^n(s) + u^n(s) u_x^n(s)) ds, \\ I_2(t) &= \int_0^t P_n g(u^n) dW^n(s). \end{aligned}$$

We will prove that I_1 is uniformly bounded in $L^2(\Omega, W^{\gamma,2}(0, T; W_\rho^{-\delta,2}(D)))$ and that I_2 is uniformly bounded in $L^2(\Omega, W^{\gamma,2}(0, T; H(D)))$ for all $0 < \gamma < \frac{1}{2}$. Let $\phi \in W_\rho^{\delta,2}(D)$, similar to (2.6), we get

$$\begin{aligned} \left| \int_{W_\rho^{-\delta,2}(D)} \langle u^n u_x^n, \phi \rangle_{W_\rho^{\delta,2}(D)} \right| &= \left| \langle u^n, u^n \phi_x \rangle_{L^2(D)} \right| \\ &\leq \|\phi_x\|_{L^\infty(D)} \|u^n\|_H^2. \end{aligned}$$

Since $2\delta > 2 + 2\alpha$, by using Lemmas 2.1-2.2, we have

$$\|\phi_x\|_{L^\infty(D)} \leq C \|\phi_x\|_{H^{\frac{1}{2}+}} \leq C \|\phi\|_{W^{\delta,2}(D)} \leq C \|\phi\|_{W_\rho^{\delta,2}(D)}.$$

Therefore, we have

$$\|u^n u_x^n\|_{W_\rho^{-\delta,2}(D)} \leq C \|u^n\|_H^2,$$

which yields that

$$\begin{aligned} \int_0^T \|I_1(t)\|_{W_\rho^{-\delta,2}(D)}^2 dt &\leq C \int_0^T \int_0^t \left(\|(-\Delta)^{\frac{\alpha}{2}} u^n(s)\|_{W_\rho^{-\delta,2}(D)}^2 + \|u^n u_x^n\|_{W_\rho^{-\delta,2}(D)}^2 \right) ds dt \\ &\leq C \int_0^T \int_0^t (\|u^n(s)\|_H^2 + \|u^n(s)\|_H^4) ds dt. \end{aligned} \quad (3.7)$$

Here we use the following fact. Let $\phi \in W_\rho^{\delta,2}(D)$. Since $(-\Delta)^{\frac{\alpha}{2}}$ is a divergence operator, we have

$$\begin{aligned} \left| \int_{W_\rho^{-\delta,2}(D)} \langle (-\Delta)^{\frac{\alpha}{2}} u, \phi \rangle_{W_\rho^{\delta,2}(D)} \right| &= \left| \langle u, (-\Delta)^{\frac{\alpha}{2}} \phi \rangle_{L^2(D)} \right| \\ &\leq \|(-\Delta)^{\frac{\alpha}{2}} \phi\|_{L^2(D)} \|u\|_H. \end{aligned} \quad (3.8)$$

Noting that $2\delta > 4 + 2\alpha$, $\delta - \frac{1}{2} > 2$ if $\alpha > 1$. It follows from Lemma 2.2 and classical Sobolev embedding theorem that $\phi \in C_0^2(D)$. We remark that we take principle value in the definition of the operator $(-\Delta)^{\frac{\alpha}{2}}$. It is easy to see that

$$\begin{aligned} (-\Delta)^{\frac{\alpha}{2}} \phi(x) &= \int_{\mathbb{R}} \frac{\phi(x+y) - \phi(x)}{|y|^{1+\alpha}} dy \\ &= \int_{\mathbb{R}} \frac{\phi(x+y) - \phi(x) - y\phi'(x)}{|y|^{1+\alpha}} dy \\ &= \int_D \phi''(\xi) |y|^{1-\alpha} dy < \infty, \end{aligned} \quad (3.9)$$

where the value of ξ is between x and y . By using (3.8) and (3.9), we can get (3.7). Moreover, using Hölder inequality and arguing as before, we obtain for $t \geq s > 0$,

$$\begin{aligned} \|I_1(t) - I_1(s)\|_{W_\rho^{-\delta,2}(D)}^2 &= \left\| \int_s^t \left(\|(-\Delta)^{\frac{\alpha}{2}} u^n(s) + u^n u_x^n ds \right) \right\|_{W_\rho^{-\delta,2}(D)}^2 \\ &\leq C(t-s) \left(\int_s^t \|u^n(r)\|_H^2 + \|u^n(r)\|_H^4 dr \right). \end{aligned} \quad (3.10)$$

Combining (3.7), (3.10) and (3.6) with $q = 4$, we have for $\gamma < \frac{1}{2}$

$$\begin{aligned} &\mathbb{E} \left(\int_0^T \|I_1(t)\|_{W_\rho^{-\delta,2}(D)}^2 dt + \int_0^T \int_0^T \frac{\|I_1(t) - I_1(s)\|_{W_\rho^{-\delta,2}(D)}^2}{|t-s|^{1+2\gamma}} dt ds \right)^{\frac{1}{2}} \\ &\leq C \mathbb{E} \left(\int_0^T \|u^n(r)\|_H^2 + \|u^n(r)\|_H^4 dr \right)^{\frac{1}{2}} \\ &\leq C < \infty \end{aligned}$$

Now, we estimate the stochastic term I_2 . Using the stochastic isometry, the contraction property of P_n and assumption (C), we have

$$\begin{aligned} \int_0^T \mathbb{E} \left\| \int_0^t P_n g(u^n(s)) dW^n(s) \right\|_H^2 dt &\leq C \int_0^T \mathbb{E} \int_0^t \|g(u^n(s))\|_2^2 ds dt \\ &\leq C \int_0^T \mathbb{E} \int_0^t (\lambda + \|u^n(s)\|_H^2) ds dt \\ &\leq C < \infty. \end{aligned}$$

For $t \geq s > 0$ and $\gamma < \frac{1}{2}$, the same ingredients above yield to

$$\begin{aligned} \mathbb{E} \int_0^T \int_0^T \frac{\|I_2(t) - I_2(s)\|_H^2}{|t-s|^{1+2\gamma}} dt ds &\leq C \mathbb{E} \int_0^T \int_0^T \frac{\int_s^t \|g(u^n(r))\|_2^2 dr}{|t-s|^{1+2\gamma}} dt ds \\ &\leq C \mathbb{E} \sup_{t \in [0,T]} (1 + \|u^n(t)\|_H^2) \int_0^T \int_0^T |t-s|^{-2\gamma} dt ds \\ &\leq C < \infty. \end{aligned}$$

The proof of the Lemma is complete. \square

Remark 3.1 *The reason we introduce the Besov-Slobodetski space is to control the term uu_x . In the ordinary case, we can not get the compact result, see the step 3.*

Step 3. Take weak limits

Lemma 3.2 [16, Theorem 2.1] *Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 in B . Let $p \in (1, \infty)$ and $\alpha \in (0, 1)$ be given. Let X be the space*

$$X = L^p(0, T; B_0) \cap W^{\gamma,p}(0, T; B_1)$$

endowed with the normal norm. Then the embedding of X in $L^p(0, T; B)$ is compact.

It follows from Lemmas 2.1 and 3.2, we have

$$W^{\gamma,2}(0, T; W_\rho^{-\delta,2}(D)) \cap L^2(0, T; V) \hookrightarrow^{compact} L^2(0, T; H).$$

Therefore, we deduce that the sequence of laws $(\mathcal{L}(u^n))_n$ is tight on $L^2(0, T; H)$. Thanks to Prokhorov's Theorem there exists a subsequence still denoted $\{u^n\}$ for which the sequence of laws $(\mathcal{L}(u^n))_n$ converges weakly in $L^2(0, T; H)$ to a probability measure μ . By using Skorokhod's embedding Theorem, we can construct a probability basis $(\Omega_*, \mathcal{F}_*, \mathbb{P}_*)$ and a sequence of $L^2(0, T; H) \cap C(0, T; W_\rho^{-\delta, 2}(D))$ -random variables $\{u_*^n\}$ and u_* such that $\mathcal{L}u_*^n = \mathcal{L}u^n$, $\forall n \in N$, $\mathcal{L}(u_*) = \mu$ and $u_*^n \rightarrow u_*$ a.s. in $L^2(0, T; H) \cap C(0, T; W_\rho^{-\delta, 2}(D))$. Moreover, $u_*^n(\cdot, \omega) \in C([0, T]; H_n)$. Thanks to Step 1 and the equality in law, we obtain that the sequence u_*^n converges weakly in $L^2(\Omega \times [0, T]; V)$ and weakly-star in $L^p(\Omega, L^\infty([0, T]; H))$ to a limit u_{**} . It is easy to verify that $u_* = u_{**}$, $dt \times dp$ -a.e. and

$$\mathbb{E}_* \sup_{t \in [0, T]} \|u_*(t)\|_H^p + \mathbb{E}_* \int_0^T \|u_*(t)\|_V^2 dt \leq C < \infty. \quad (3.11)$$

We introduce the filtration

$$(\mathcal{F}_n^*)_t := \sigma\{u_*^n(s), s \leq t\},$$

and construct (w.r.t. $(\mathcal{F}_n^*)_t$) the time continuous square integrable martingale $(M_n(t), t \in [0, T])$ with trajectories in $C([0, T; H])$ by

$$M_n(t) := u_*^n(t) - P_n u_0 + \int_0^t (-\Delta)^{\frac{\alpha}{2}} u_*^n(s) ds - \int_0^t u_*^n(s) (u_*^n(s))_x ds.$$

The equality in law yields to the fact that the quadratic variation is given by

$$\ll M_n \gg_t = \int_0^t P_n g(u_*^n(s)) Q g(u_*^n(s))^* ds,$$

where $g(u_*^n(s))^*$ is the adjoint of $g(u_*^n(s))$. We will prove $M_n(t)$ converges weakly in $W_\rho^{-\delta, 2}(D)$ to the martingale $M(t)$ for all $t \in [0, T]$, where $M(t)$ is given by

$$M(t) := u_*(t) - u_0 + \int_0^t (-\Delta)^{\frac{\alpha}{2}} u_*(s) ds - \int_0^t u_*(s) (u_*(s))_x ds.$$

It follows from Lemma 3.1 that $\|M_n(t)\|_{W_\rho^{-\delta, 2}(D)} \leq C < \infty$, where C does not depend on n . Since $W_\rho^{-\delta, 2}(D)$ is a Hilbert space, we have $M_n(t) \rightharpoonup M(t)$, as $n \rightarrow \infty$, $t \in [0, T]$. Now we apply to the representation Theorem [10, Theorem 8.2], we infer that there exists a probability basis $(\Omega^*, \mathcal{F}^*, \mathbb{R}^*, W^*)$ such that

$$M(t) = \int_0^t g(u_*) dW^*(s).$$

By using Burkholder-Davis-Gundy inequality and (3.11), we have

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t g(u_*(s)) dW^*(s) \right\|_H^2 &\leq C \mathbb{E} \int_0^T \|g(u_*(s))\|_2^2 ds \\ &\leq C(1 + \mathbb{E} \sup_{t \in [0, T]} \|u_*(s)\|_H^2) < \infty. \end{aligned}$$

Furthermore, by using Lemmas 2.1-2.2, (2.6) and $1 < \alpha < 2$, we get

$$\begin{aligned} &\mathbb{E} \int_0^T (\|(-\Delta)^{\frac{\alpha}{2}} u_*(s)\|_{W_\rho^{-\frac{\alpha}{2}, 2}(D)} + \|(u_*^2(s))_x\|_{W_\rho^{-\frac{\alpha}{2}, 2}(D)}) ds \\ &\leq C \mathbb{E} \int_0^T (\|u_*(s)\|_V + \|u_*^2(s)\|_{W_\rho^{1-\frac{\alpha}{2}, 2}(D)}) ds \\ &\leq C \mathbb{E} \int_0^T (1 + \|u_*(s)\|_V) ds < \infty, \end{aligned}$$

where we used the facts $u_*(s) \in C(\bar{D})$ because of $\alpha > 1$, and

$$\|(u_*^2(s))_x\|_{W_\rho^{-\frac{\alpha}{2},2}(D)} \leq C \|u_*^2(s)\|_{W_\rho^{1-\frac{\alpha}{2},2}(D)}.$$

By using Fourier transform, one can prove that the above inequality holds for $D = \mathbb{R}$. Noting that $u_* \equiv 0$ in $\mathbb{R} \setminus D$, we have the above inequality holds. Actually, even if u_* does not define in $\mathbb{R} \setminus D$, we can also get the desire result under the condition that D is an *extension domain*. Because the domain D is an *extension domain*, we can extend u to \mathbb{R} by letting $u = 0$ in $\mathbb{R} \setminus D$. We denote it by \tilde{u} , and obtain that

$$\|u\|_{W_\rho^{-\frac{\alpha}{2},2}(D)} \leq C \|\tilde{u}\|_{W_\rho^{1-\frac{\alpha}{2},2}(\mathbb{R})} \leq C \|u\|_{W_\rho^{1-\frac{\alpha}{2},2}(D)}.$$

Using the densely embedding $W_\rho^{\delta,2}(D) \hookrightarrow V$, we conclude that (3.1) holds in the $W_\rho^{\frac{\alpha}{2},2}(D)$ - $W_\rho^{-\frac{\alpha}{2},2}(D)$ -duality. This complete the proof of Theorem 3.1. \square

Remark 3.2 *We can use the same method to deal with the following problem*

$$\begin{cases} du_t = \left(-(-\Delta)^{\frac{\alpha}{2}}u - uu_x\right) dt + g(u)dW_t, & t > 0, x \in D, \\ u(x, t)|_{D^c} = g(x, t), \\ u(x, 0) = u_0(x), \end{cases} \quad (3.12)$$

where $D = (0, 1)$. By letting $v = u - g$, we can obtain the existence of martingale solution of (3.12) under the suitable assumption on g .

4 Weak solution for a deterministic nonlocal Burgers equation

In this section, we will consider the corresponding deterministic version of the equation (1.1) considered in the previous section. That is, we consider the following deterministic nonlocal Burgers equation on a bounded interval

$$\begin{cases} u(t) + (-\Delta)^{\frac{\alpha}{2}}u + uu_x = 0, & t > 0, x \in D, \\ u|_{D^c} = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (4.1)$$

where $D = (-1, 1)$ and $D^c = \mathbb{R}^1 \setminus D$.

In section 3, we obtained the existence of martingale solution to (1.1). Unfortunately, we can not get the uniqueness of the martingale solution. We are unaware of an existence of weak solution result about the nonlocal Burgers equation (4.1) on a bounded domain. On the whole space $D = \mathbb{R}^1$, there are a lot of results for (4.1); see, for example, [3, 4, 5, 6].

In the following, we will adopt the similar method to section 3 to prove the existence of L^2 -solution of (4.1). Firstly, we give the definition of L^2 -solution. We will adopt the same symbol as in section 2. Let

$$C_\rho^n(D) := \left\{ u \in C^n(D), \quad \rho(x)u^{(n)}(x) \in L^\infty(D) \right\}.$$

Definition 4.1 *We say that u is a weak solution of (4.1) if $u \in L^\infty(0, T; L^2(D))$ for each $T > 0$, such that u satisfies*

$$(u, \phi) + \int_0^t (u, (-\Delta)^{\alpha/2} \phi) ds - \frac{1}{2} \int_0^t (u^2, \phi_x) ds = (u_0, \phi),$$

for each $\phi \in C_\rho^2(D)$.

Theorem 4.1 For $u_0 \in L^2(D)$, there exists a weak solution to equation (4.1).

Proof. We will use a Galerkin approximation and Lemma 3.2 to prove this Theorem. Similar to the proof of Theorem 3.1, let

$$\{e_1, e_2, \dots\} \subset V$$

be an orthonormal basis of H and let $H_n := \text{span}\{e_1, \dots, e_n\}$ such that $\{e_1, e_2, \dots\}$ is dense in V . Let $P_n : V^* \mapsto H_n$ be defined by

$$P_n y := \sum_{i=1}^n \langle y, e_i \rangle e_i, \quad y \in V^*.$$

Obviously, $P_n|_H$ is just the orthogonal projection onto H_n in H and we have

$${}_{V^*} \langle P_n(-\Delta)^{\frac{\alpha}{2}} u, v \rangle_V = \langle P_n(-\Delta)^{\frac{\alpha}{2}} u, v \rangle_H = {}_{V^*} \langle (-\Delta)^{\frac{\alpha}{2}} u, v \rangle_V, \quad u \in V, \quad v \in H_n,$$

where ${}_{V^*} \langle \cdot, \cdot \rangle_V$ denotes the dualization between V and its dual space V^* . Then for each finite $n \in \mathbb{N}$, we consider the following stochastic equation on H_n

$$\begin{cases} \frac{du^n(t)}{dt} = -P_n(-\Delta)^{\alpha/2} u^n(t) - P_n(u^n u_x^n), & t \in [0, T], \\ u^n(0) = P_n u_0 = u_0^n. \end{cases} \quad (4.2)$$

Since the finite dimensional space stochastic differential equation (4.2) has locally Lipschitz and linear growth coefficient, the equation (3.2) admits a unique strong solution $(u^n(t) \in L^2(\Omega; C([0, T]; H_n)))$.

Multiplying (4.2) by u^n and integrating over $D \times [0, t]$, we have

$$\begin{aligned} \|u^n(t)\|_H^2 &= \|u_0^n\|_H^2 - \int_0^t ((-\Delta)^{\frac{\alpha}{2}} u^n(s), u^n(s))_H ds \\ &= \|u_0^n\|_H^2 - \int_0^t \|u^n(s)\|_V^2 ds, \end{aligned} \quad (4.3)$$

where we have used the facts $(u^n u_x^n, u^n) = 0$ and (2.4). Equality (4.3) implies that $\|u^n\|_H^2 \leq \|u_0\|_H^2$, which yields that there exists a subsequence of $\{u^n\}$, still denoted $\{u^n\}$, such that $u^n \rightharpoonup u$ as $n \rightarrow \infty$, where $u \in H$.

Next we prove that the sequence $\{u^n\}$ is uniformly bounded in the space $W^{\gamma, 2}(0, T; W_{\rho}^{-\delta, 2}(D)) \cap L^2(0, T; V)$. Inequality (4.3) implies that $\{u^n\}_{n=1, 2, \dots}$ is uniformly bounded in $L^2(0, T; V)$. As $\{u^n(t)\}_{t \in [0, T]}$ is the strong solution of the finite dimensional stochastic differential equation (3.2), then $u^n(t)$ is the solution of the stochastic integral equation

$$u^n(t) = P_n u_0 - \int_0^t ((-\Delta)^{\alpha/2} u^n(s) + u^n(s) u_x^n(s)) ds$$

for all $t \in [0, T]$. We denote by

$$I(t) = - \int_0^t ((-\Delta)^{\alpha/2} u^n(s) + u^n(s) u_x^n(s)) ds.$$

It is remarked that $\sup_{t \in [0, T]} \|u^n(t)\|_H^2 \leq C$ implies that $\sup_{t \in [0, T]} \|u^n(t)\|_H^4 \leq C$. Similar to the proof of Lemma 3.1, one can prove that

$$\int_0^T \|I(t)\|_{W_{\rho}^{-\delta, 2}(D)}^2 dt + \int_0^T \int_0^T \frac{\|I(t) - I(s)\|_{W_{\rho}^{-\delta, 2}(D)}^2}{|t - s|^{1+2\gamma}} dt ds \leq C.$$

It follows from Lemmas 2.1 and 3.2, we have

$$W^{\gamma,2}(0,T;W_{\rho}^{-\delta,2}(D)) \cap L^2(0,T;V) \hookrightarrow^{compact} L^2(0,T;H).$$

Therefore, we deduce that the sequence $\{u^n\}$ converges to some u^* in $L^2(0,T;H)$. Due to the uniqueness of the limit, we obtain that $u = u^*$. Let $\phi \in C_{\rho}^2(D)$, then we have

$$(u^n, \phi) + \int_0^t (u^n, (-\Delta)^{\alpha/2} \phi) ds - \frac{1}{2} \int_0^t ((u^n)^2, \phi_x) ds = (u_0^n, \phi).$$

Let $n \rightarrow \infty$, we have

$$(u, \phi) + \int_0^t (u, (-\Delta)^{\alpha/2} \phi) ds - \frac{1}{2} \int_0^t (u^2, \phi_x) ds = (u_0, \phi).$$

This completes the proof. \square

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